The principal mathematical theorem of the Great Pyramid of Giza

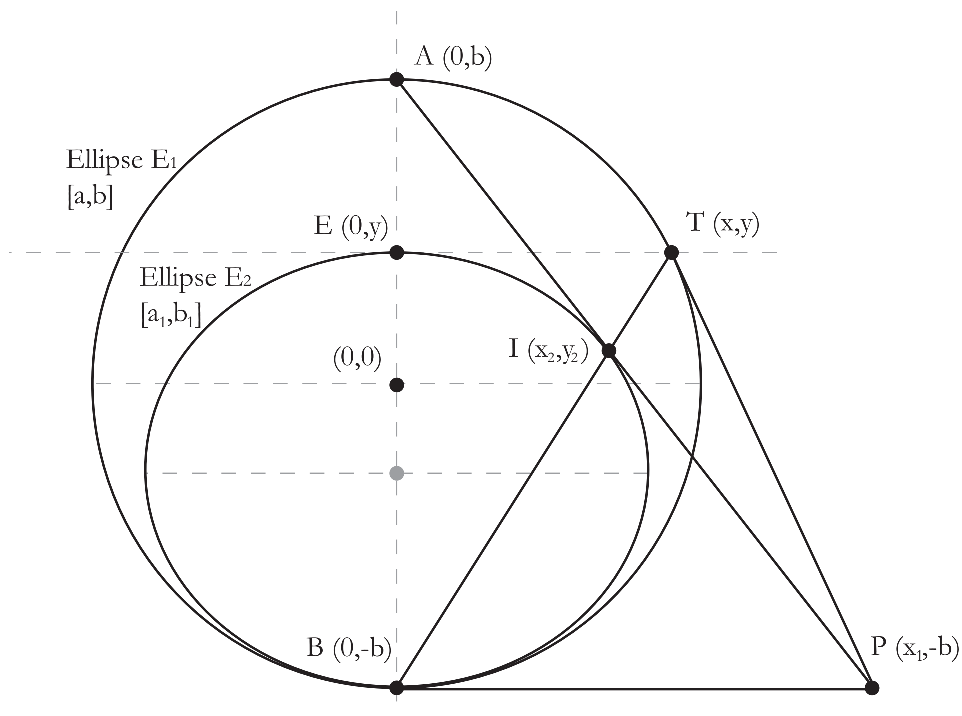
Lemma 2 The exit points of the upper shaftways of the pyramid are positioned in accordance with the following mathematics theorem:

Let E_1 be an ellipse with standard equation $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Let $A(0, b)$ and $B(0,-b)$ be the two ends of the ellipse's minor axis. Let the point $T(x, y)$ be the tangent point on the ellipse for a line emanating from the point $P(x_1,-b)$.

Let E_2 be an ellipse with standard equation $\frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} = 1$ which has the line PA as a tangent and whose minor axis has the end points $B(0,-b)$ and $E(0, y)$.

Then the intersect point of the lines PA and BT, $I(x_2, y_2)$, is also the tangent point of the ellipse E_2 with the line PA.



Proof : As the geometric system is dependent primarily upon the positioning of the point $P(x_1,-b)$ then all calculations will be in reference to this point. In order to prove the lemma, a) the steps required to calculate the intersect point of the lines PA and BT, $I(x_2, y_2)$ will be looked at first b) those to calculate the tangent point of PA and E_2 will be considered second and c) how the numeric values of the algebraic equations relate to the pyramid's shaftways will be considered third.

(a) The intersect point of the lines *PA* and *BT*

1) We first need to determine the algebraic coordinates of the tangent point *T*.

The tangent to an ellipse at the point (x, y) can be determined by differentiating the standard formula for an ellipse as an implicit function :

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{2.0}$$

$$\frac{2x}{a^2} + \frac{2y\frac{\delta y}{\delta x}}{b^2} = 0$$

$$\frac{\delta y}{\delta x} = \frac{-xb^2}{ya^2} \tag{2.1}$$

On the diagram this algebraic expression represents the gradient of the line *PT*, which is also given by the formula for the gradient of a straight line between two points *(x, y)* and *(x1, y1)*

$$line\ gradient = \frac{y - y_1}{x - x_1} \tag{2.2}$$

In this particular case $y_1 = -b$ by definition of the point P, and so equations 2.1 and 2.2 can be solved simultaneously giving

$$\frac{-xb^2}{ya^2} = \frac{y+b}{x-x_1}$$
$$-xb^2(x - x_1) = ya^2(y+b)$$
$$-x^2b^2 + xx_1b^2 = y^2a^2 + ya^2b \tag{2.3}$$

Re-working the standard equation of an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
$$y^2 = \frac{b^2(a^2 - x^2)}{a^2}$$
$$y = \frac{b\sqrt{a^2 - x^2}}{a}$$

and substituting this value of y into equation 2.3 gives

$$-x^2b^2 + xx_1b^2 = \frac{b^2(a^2 - x^2)a^2}{a^2} + \frac{b\sqrt{a^2 - x^2}.a^2b}{a}$$
$$-x^2b^2 + xx_1b^2 = b^2(a^2 - x^2) + ab^2\sqrt{a^2 - x^2}$$
$$a\sqrt{a^2 - x^2} = xx_1 - a^2$$
$$a^2(a^2 - x^2) = x^2x_1^2 + a^4 - 2xx_1a^2$$
$$a^4 - a^2x^2 - x^2x_1^2 - a^4 + 2xx_1a^2 = 0$$
$$x^2(x_1^2 + a^2) + x(-2x_1a^2) = 0$$

This equation is now in the format where it can be solved using the standard formula for solving quadratic equations, giving

$$x = \frac{2x_1a^2 \pm \sqrt{4x_1^2a^4 - 4(x_1^2 + a^2).0}}{2(x_1^2 + a^2)}$$
$$x = \frac{2x_1a^2 \pm 2x_1a^2}{2(x_1^2 + a^2)}$$
$$x = \frac{2x_1a^2}{x_1^2 + a^2} \quad or \quad x = 0 \tag{2.4}$$

This is the algebraic value of the *x* coordinate of the tangent point, and to determine the *y* coordinate we can substitute the non zero value of *x* back into the standard ellipse equation, giving

$$\frac{\frac{4x_1^2a^4}{(x_1^2 + a^2)^2}}{a^2} + \frac{y^2}{b^2} = 1$$
$$\frac{4x_1^2a^2}{(x_1^2 + a^2)^2} + \frac{y^2}{b^2} = 1$$
$$y^2 = b^2 - \frac{4x_1^2a^2b^2}{(x_1^2 + a^2)^2}$$
$$y^2 = \frac{b^2(x_1^2 + a^2)^2 - 4x_1^2a^2b^2}{(x_1^2 + a^2)^2}$$
$$y^2 = \frac{b^2x1^2 + b^2a^4 + 2x1^2a^2b^2 - 4x_1^2a^2b^2}{(x_1^2 + a^2)^2}$$
$$y = \frac{b(x_1^2 - a^2)}{x_1^2 + a^2} \tag{2.5}$$

Therefore the tangent point T(x,y) has the algebraic coordinates

$$T\left(\frac{2x_1a^2}{x_1^2 + a^2}, \frac{b(x_1^2 - a^2)}{x_1^2 + a^2}\right) \tag{2.6}$$

2) Second we need to determine the equations of the lines *BA* and *BT* and thereby determine the their intersect point.

The gradient of the line BT can be calculated by substituting the known end points of the line into equation 2.2 where

$$y_1 = -b$$
$$x_1 = 0$$
$$y = \frac{b(x_1^2 - a^2)}{x_1^2 + a^2}$$
$$x = \frac{2x_1a^2}{x_1^2 + a^2}$$

giving

$$gradient\ BT = \frac{\dfrac{b(x_1^2 - a^2)}{x_1^2 + a^2} + b}{\dfrac{2x_1 a^2}{x_1^2 + a^2} - 0}$$

$$gradient\ BT = \frac{b(x_1^2 - a^2) + b(x_1^2 + a^2)}{2x_1 a^2}$$

$$gradient\ BT = \frac{bx_1}{a^2}$$

The equation of the line *BT* is therefore

$$y = \frac{bx_1 x}{a^2} - b \tag{2.7}$$

The end points of the line *PA* are $(x_1, -b)$ and $(0, b)$, and substituting these into equation 2.2 gives the line gradient as

$$gradient\ PA = \frac{-2b}{x_1}$$

so the equation of the line *PA* is

$$y = \frac{-2bx}{x_1} + b \tag{2.8}$$

The intersect point can now be found by simultaneously solving the two line equations 2.7 and 2.8 giving the point $(x2, y2)$

$$\frac{bx_1 x_2}{a^2} - b = \frac{-2bx_2}{x_1} + b$$

$$\frac{bx_1^2 x_2 + 2bx_2 a^2}{a^2 x_1} = 2b$$

$$x_2 (x_1^2 + 2a^2) = 2a^2 x_1$$

$$x_2 = \frac{2a^2 x_1}{x_1^2 + 2a^2}$$

substituting this value of x_2 into the equation 2.7 gives

$$y_2 = \frac{bx_1 2a^2 x_1}{a^2 (x_1^2 + 2a^2)} - b$$

$$y_2 = \frac{2bx_1^2 - b(x_1^2 + 2a^2)}{x_1^2 + 2a^2}$$

$$y_2 = \frac{b(x_1^2 - 2a^2)}{x_1^2 + 2a^2}$$

The intersect point of the lines *PA* and *BT*, $I(x_2, y_2)$ is therefore

$$I\left(\frac{2a^2 x_1}{x_1^2 + 2a^2}, \frac{b(x_1^2 - 2a^2)}{x_1^2 + 2a^2}\right) \tag{2.9}$$

(b) The tangent point of line PA and ellipse E_2

In order to find the tangent point of PA with E_2 we need to 1) find the general solution to a line-ellipse tangent point, 2) transform the line PA into the coordinate system of ellipse E_2, 3) calculate the radii of ellipse E_2 in terms of the radii of ellipse E_1 and 4) solve the general line-ellipse tangent equation using the results of 2) and 3).

1) The general solution to a line-ellipse tangent point.

Re-working the general equation of an ellipse, 2.0, gives

$$y^2 = \frac{b^2(a^2 - x^2)}{a^2}$$

For the general formula for a straight line

$$\begin{aligned} y &= mx + c \\ y^2 &= m^2x^2 + 2mxc + c^2 \end{aligned} \tag{2.10}$$

so, at the intersect(s) between a straight line and ellipse

$$m^2x^2 + 2mxc + c^2 = \frac{b^2(a^2 - x^2)}{a^2}$$

$$m^2x^2a^2 + 2mxca^2 + a^2c^2 = b^2a^2 - b^2x^2$$

$$x2(b2 + m2a2) + x(2mca2) + a2(c2 - b2) = 0$$

and solving this using the standard solution for quadratic equations gives

$$x = \frac{-2mca^2 \pm \sqrt{4m^2c^2a^4 - 4(b^2 + m^2a^2)(a^2(c^2 - b^2))}}{2(b^2 + m^2a^2)}$$

$$x = \frac{-2mca^2 \pm \sqrt{4m^2c^2a^4 - 4a^2b^2c^2 - 4m^2a^4c^2 + 4a^2b^4 + 4m^2a^4b^2}}{2(b^2 + m^2a^2)}$$

$$x = \frac{-mca^2 \pm \sqrt{a^2b^2(b^2 + m^2a^2 - c^2)}}{b^2 + m^2a^2}$$

At the tangent point, where there must be only one solution for x and therefore the $\pm$ term must be zero

$$x = \frac{-mca^2}{m^2a^2 + b^2} \tag{2.11}$$

and substituting this value of x into the general straight line equation 2.10 gives

$$y = \frac{-m^2ca^2}{m^2a^2 + b^2} + c^2$$

$$y = \frac{cb^2}{m^2a^2 + b^2} \tag{2.12}$$

2) Transform the line PA into the coordinate system of ellipse E_2.

From the diagram, if the ellipse E_2 has radii of a_1 and b_1 then the distance d between the centres of the ellipses E_1 and E_2 is

$$d = b - b1$$

and by definition

$$2b_1 = b + y$$

The value of y in terms of x_1 was calculated earlier and shown in equation 2.5, so substituting this values into the equation above gives

$$2b_1 = b + \frac{b(x_1^2 - a^2)}{x_1^2 + a^2}$$
$$2b_1 = \frac{b(x_1^2 + a^2) + b(x_1^2 - a^2)}{x_1^2 + a2}$$
$$b_1 = \frac{bx_1^2}{x_1^2 + a^2} \tag{2.13}$$

therefore the distance d is given by

$$d = b - \frac{bx_1^2}{x_1^2 + a^2}$$
$$d = \frac{b(x_1^2 + a^2) - bx_1^2}{x_1^2 + a^2}$$
$$d = \frac{ba^2}{x_1^2 + a^2}$$

The equation of the line PA in the primary coordinate system is given in equation 2.8, and adding the value of d from above to re-base the line to a coordinate system centred on ellipse E_2 gives the line equation as

$$y = \frac{-2bx}{x_1} + b + \frac{ba^2}{x_1^2 + a^2}$$
$$y = \frac{-2bx}{x_1} + \frac{b(x_1^2 + a^2) + ba^2}{x_1^2 + a^2}$$
$$y = \frac{-2bx}{x_1} + \frac{b(x_1^2 + 2a^2)}{x_1^2 + a^2} \tag{2.14}$$

3) Calculate the radii of ellipse E_2 in terms of the radii of ellipse E_1

The algebraic value of the minor radius of ellipse E_2 in terms of the radii of E_1 has already been shown in equation 2.13. Equation 2.6 gave the tangent point $T(x, y)$ on the ellipse E_1, and by the same argument the tangent point $I(x_2, y_2)$ based to the coordinate system of ellipse E_2 must be

$$T\left(\frac{2x_1 a_1^2}{x_1^2 + a_1^2}, \frac{b_1(x_1^2 - a_1^2)}{x_1^2 + a_1^2}\right)$$

therefore the gradient of the line PA, by substituting the points P and I into equation 2.2 is

$$gradient\ PI = \frac{\dfrac{b(x_1^2 - a_1^2)}{x_1^2 + a_1^2} + b_1}{\dfrac{2x_1 a_1^2}{x_1^2 + a_1^2} - x_1}$$

$$gradient\ PI = \frac{b_1(x_1^2 - a_1^2) + b_1(x_1^2 + a_1^2)}{2x_1 a_1^2 - x_1(x_1^2 + a_1^2)}$$

$$gradient\ PI = \frac{2b_1 x_1}{a_1^2 - x_1^2}$$

and substituting the value of b_1 from equation 2.13 into the above equation gives

$$gradient\ PI = \frac{2bx_1^2 x_1}{(a_1^2 - x_1^2)(x_1^2 + a^2)}$$

This gradient must be equal to the gradient of the line PA in the transformed coordinate system, and shown in equation 2.14, so

$$\frac{-2b}{x_1} = \frac{2bx_1^2 x_1}{(a_1^2 - x_1^2)(x_1^2 + a_1^2)}$$

$$-(a_1^2 - x_1^2)(x_1^2 + a_1^2) = x_1^4$$

$$-a_1^2 a^2 - a_1^2 x_1^2 + x_1^2 a^2 + x_1^4 = x_1^4$$

$$x_1^2 a^2 = a_1^2 a^2 + a_1^2 x_1^2$$

$$a_1^2 = \frac{x_1^2 a^2}{x_1^2 + a^2}$$

$$a_1 = \frac{ax_1}{\sqrt{x_1^2 + a^2}} \qquad (2.15)$$

4) Solve the general line-ellipse tangent equation using the results of 2) and 3) above.

The x co-ordinate of the tangent point of any line with any ellipse is given in equation 2.11 and we can customise this to ellipse E_2 to give

$$x_2 = \frac{-mca_1^2}{m^2 a_1^2 + b^2}$$

we can substitute into this equation the values just calculated of

$$a_1^2 = \frac{a^2 x_1^2}{x_1^2 + a^2}$$

$$b_1^2 = \frac{b^2 x_1^4}{(x_1^2 + a^2)^2}$$

$$m = \frac{-2b}{x_1}$$

$$c = \frac{b(x_1^2 + 2a^2)}{x_1^2 + a^2}$$

therefore

$$x_2 = \frac{\frac{2b}{x_1} \cdot \frac{b(x_1^2 + 2a^2)}{x_1^2 + a^2} \cdot \frac{a^2 x_1^2}{x_1^2 + a^2}}{\frac{4b^2}{x_1^2} \cdot \frac{a^2 x_1^2}{x_1^2 + a^2} + \frac{b^2 x_1^4}{(x_1^2 + a^2)^2}}$$

$$x_2 = \frac{\frac{2b^2 a^2 x_1 (x_1^2 + 2a^2)}{(x_1^2 + a^2)^2}}{\frac{4b^2 a^2 (x_1^2 + a^2) + b^2 x_1^4}{(x_1^2 + a^2)^2}}$$

$$x_2 = \frac{2a^2 x_1 (x_1^2 + 2a^2)}{4a^2 x_1^2 + 4a^4 + x_1^4}$$

$$x_2 = \frac{2a^2 x_1}{x_1^2 + 2a^2}$$

Substituting this value of x_2 into the original equation 2.8 of line PA gives

$$y_2 = \frac{-2b2a^2 x_1}{x_1 (x_1^2 + 2a^2)} + b$$

$$y_2 = \frac{-4ba^2 + b(x_1^2 + 2a^2)}{x_1^2 + 2a^2}$$

$$y_2 = \frac{b(x_1^2 - 2a^2)}{x_1^2 + 2a^2}$$

The tangent point of the line *PA* with ellipse E_2 at the point $I(x_2, y_2)$ is

$$I\left(\frac{2a^2 x_1}{x_1^2 + 2a^2}, \frac{b(x_1^2 - 2a^2)}{x_1^2 + 2a^2}\right) \tag{2.16}$$

and it can be seen that 2.16 and 2.9 are identical.

(b) Comparing the principle theorem to the pyramid's upper shaft

To compare the geometric design to the architectural design we need to 1) evaluate the numerical values of the algebraic equations and then 2) compare these values to the surveyed points in the pyramid's architecture.

1) The numerical values of the algebraic equations

The tangent point of ellipse E_2 with the line PA was given in equation 2.16. To enumerate this equation for I_x, the algebraic value of a from equation 1.4 can be substituted into the equation giving

$$I_x = \frac{\dfrac{2.16x_1^2 f^2 \,.\, x_1}{\pi^2 R_c^2}}{x_1^2 + \dfrac{2.16x_1^2 f^2}{\pi^2 Rc^2}}$$

$$I_x = \frac{32x_1^3 f^2}{\pi^2 R_c^2 \left(x_1^2 + \dfrac{32x_1^2 f^2}{\pi^2 Rc^2} \right)}$$

$$I_x = \frac{32x_1 f^2}{\pi^2 R_c^2 + 32f^2}$$

The numerical values that we have already calculated for the ellipse of the Earth and for the scaled model of the Earth are as follows

$$f = 1.00336409$$

$$R_c = 2.003365502$$

$$x_1 = 115.3664\, m$$

and so the value of I_x is

$$I_x = 51.7438\, m$$

The value of the y coordinate, I_y, can be found in a similar manner by substituting the values of a and b from equations 1.3 and 1.4 into the algebraic value of I_y given in equation 2.16 giving

$$I_y = \frac{\dfrac{4x_1}{\pi R_c}\left(x_1^2 - 2.\dfrac{16x_1^2 f^2}{\pi^2 R_c^2} \right)}{x_1^2 + 2.\dfrac{16x_1^2 f^2}{\pi^2 R_c^2}}$$

$$I_y = \frac{4x_1}{\pi R_c}\left(\frac{x_1^2 \pi^2 R_c^2 - 32x_1^2 f^2}{\pi^2 R_c^2} \right)\left(\frac{\pi^2 R_c^2}{x_1^2 \pi^2 R_c^2 + 32x_1^2 f^2} \right)$$

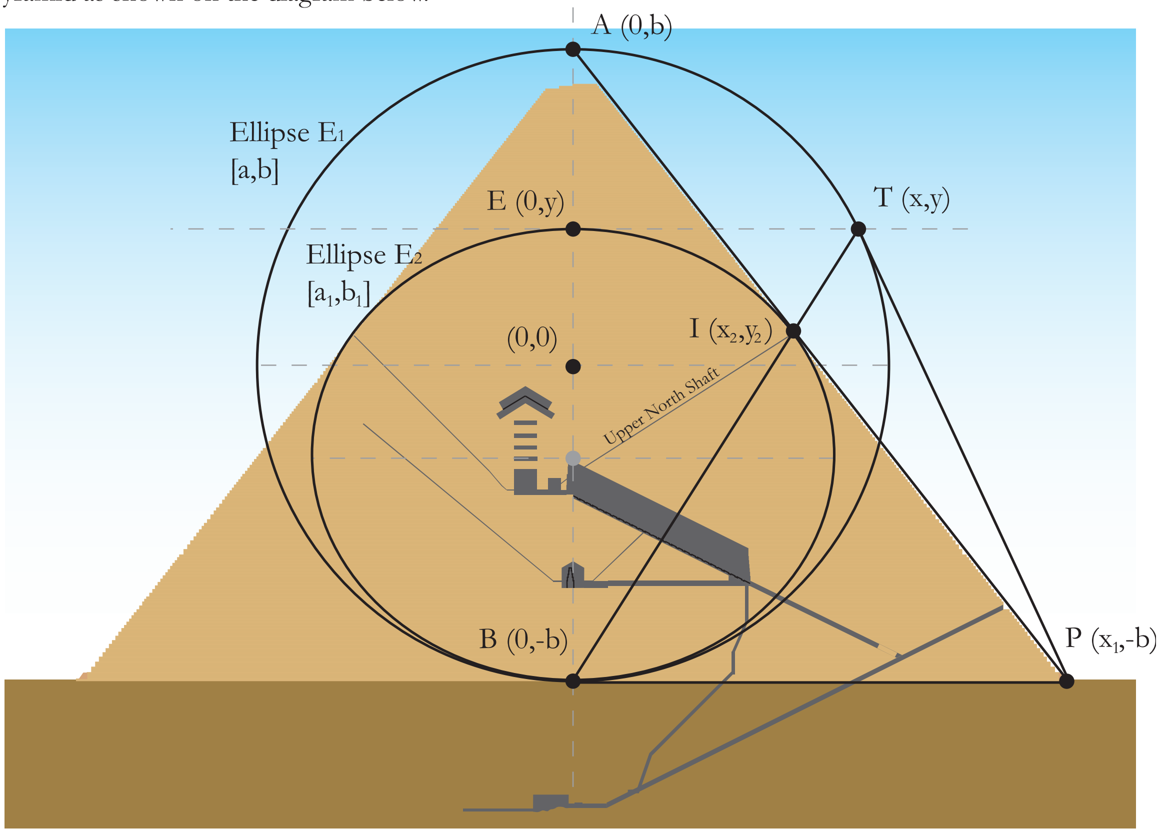
$$I_y = \frac{4x_1 (\pi^2 R_c^2 - 32f^2)}{\pi R_c (\pi^2 R_c^2 + 32f^2)}$$

$$Iy = 7.5496\, m$$

so from the algebraic calculations

$$I(x2,y2) = (51.7438, 7.5496) \tag{2.17}$$

2) The numerical values from the surveying data need to be extracted from the upper northern shaft of the pyramid as shown on the diagram below.



According to the lemma 2 this shaft emerges from the pyramid at the point $I(x_2,y_2)$ and since the external stonework around the shaft's exit is largely destroyed, it is necessary to use the straight shaft roof line as a reference, for which we will need its mathematical line equation.

The surveying data for this section of the pyramid comes from Rudolf Gantenbrink[6] from which we can extract two arbitrarily chosen points on the shaft's roof. Since Gantenbrink used the base center as his coordinate center point, and we are using the ellipse center, the y value from the survey needs rebasing to our coordinate system by subtracting the value of the ellipse's semi-minor axis of 73.3211m

	x coordinate	y coordinate	y coordinte transformed
Block 7 upper end	5.6016	51.4472	-21.8740
Block 25 upper end	39.5447	73.1544	-0.1667

6. *Gantenbrink, Rudolf The Upuaut Project, www.cheops.org (cyber drawings - cheops shafts) Original DWF file*

The gradient and angle of the shaft's roof are therefore

$$gradient\ SHAFT = \frac{-0.1667 + 21.8740}{39.5447 - 5.6016}$$
$$gradient\ SHAFT = 0.6395$$
$$angle\ SHAFT = 32.599635^o$$

and thereby the constant in the line equation of the shaft's roof is

$$-0.1667 = 0.6395 \text{ x } 39.5447 + c$$
$$c = -25.4563$$

and the equation of the shaft roof

$$y = 0.6395x - 25.4563$$

The equation of the pyramid's northern face, the line *PA*, can be evaluated from equation 2.8 giving

$$y = -1.2711x + 73.3211$$

and solving these two equations to find the shaft's intersect point with the face gives

$$0.6385x - 25.4563 = -1.2711x + 73.3211$$
$$1.9096x = 98.7774$$
$$x = 51.7267$$

and thereby the value of y as

$$y = -1.2711 \text{ x } 51.7267 + 73.3211$$
$$y = 7.5713$$

and so the surveying data gives the shaft exit point as being

$$I(x2, y2) = (51.7267, 7.5713) \qquad (2.18)$$

Comparing the points from the algebraic calculations (2.17) and those from the surveying (2.18) shows discrepancies of 0.0171m on the x coordinte and 0.0217m on the y coordinate.

These discrepancies are sufficiently small to be able to state that lemma 2 is proven.